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Classical Option Pricing and Some Steps Further

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ABSTRACT

This paper considers the asset price p as relations $C=pV$ between the value C and the volume V of the executed transactions and studies the consequences for the option pricing equations. We show that the classical BSM model implicitly assumes that value C and volume V of transactions follow identical Brownian processes. Violation of this identity leads to 2-dimensional BSM-like equation with two constant volatilities. We show that agents expectations can increase the dimension of the BSM model. We study the case when agents expectations may depend on the option price data and show that such assumption can lead to the nonlinear BSM-like equations. We reconsider the Heston stochastic volatility model for the price determined by the value and the volume and derive 3-dimensional BSM-like model with stochastic value volatility and constant volume volatility. Variety of the BSM-like equations states the problem of reasonable balance between the accuracy and the complexity of the option pricing equations.

Keywords: Option Pricing, Black-Scholes-Merton Equations, Stochastic Volatility, Market Transactions, Expectations, nonlinear equations

JEL : C10, G110, G120, G130

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1. Introduction.

We consider classical option pricing (Black and Scholes 1973; Merton 1973) (BSM) non-linear and stochastic volatility (Heston 1993) models and suggest some extensions. Classical papers by Black, Scholes (1973) and Merton (1973) were published almost 50 years ago and nevertheless their results impact current development and further research of assets and option pricing models. Hundreds researches contribute to this important financial problems. We refer only few papers by Bates (1996), Merton (1997), Scholes (1997) and some who contribute to general treatment of options pricing and financial methods - Figlewski (1998), Shiryaev (1999), Hull (2009). Many researchers provide extension of the BSM model: Black, Derman and Toy (1990), Hull and White (2001). Extensions of BSM from constant to stochastic volatility were developed by Hull and White (1987), Heston (1993), Ball and Roma (1994), Saikat (1996), Britten-Jones and Neuberger (2000), Engle and Figlewski (2014), Cohen and Tegner (2018), multiple assets option pricing models by Broadie and Detemple (1997), Rapuch and Roncalli (2004), Carmona and Durrleman (2006), Li, Deng and Zhou, (2010), application of Non-Gaussian processes by Borland (2004), extension of diffusion by Kleinert and Korbel (2016). Some collections of different approaches to BSM model are presented in Choi (2018). We have no intend to give any reasonable review of current state of derivatives and option pricing theory. Instead we indicate some directions for classical BSM model extensions – extension from constant to stochastic volatility models, multiple assets options pricing, non-Brownian random processes and etc. It seems that during these 50 years after Black and Scholes (1973), and Merton (1973) studies almost all possible methods and directions for options pricing modeling are described.

Nevertheless we regard the classical BSM model as an endless source for further development. Time by time it is useful to state a simple questions to the classical models. It may help to find out the way for further progress. In this paper we consider the classical BSM model and state a simple question – what economic factors define price evolution and how that correspond with the assumption on Brownian behavior of the underlying asset's price?

Indeed, the asset price is not the March Hare from Lewis Carroll's "Alice's Adventures in Wonderland" that can jump randomly. The price is not a stand-alone financial notion and don't behave arbitrary like "The Cat that walked by himself" by R. Kipling. The price dynamics is determined by numerous economic and financial factors. Stochastic behavior of these factors impact stochastic behavior of the price. Option pricing models should

correspond the general relations between economic and financial variables and market transactions that determine random market evolution and define stochastic price dynamics. Otherwise some studies on option pricing seem alike to math gambling to guess the correct form for the stochastic process that govern the price or its volatility without efforts to understand financial reasons of such dynamics. Numerous such attempts improve classical BSM model and enhance the option price studies but give a little for the understanding – what are the financial relations that govern the assets price randomness?

This paper use common price definition as result of executed market transactions. Researchers use numerous definitions of price. More than a century ago Fetter (1912) mentioned 117 price definition and one of them define the price as “Ratio-of-exchange definitions of price in terms of value in the sense of a mere ratio of exchange”. In this paper we use this particular price definition as the only tool for description of price random properties and option pricing. This price notion is well known and defines the price p of the transaction as coefficient between the total value C and the total volume V of the transaction:

$$C = pV \quad (1.1)$$

It is obvious that other numerous price definitions due to Fetter (1912) and their usage for modeling or divinations of asset price dynamics, as endogenous or exogenous price process will remain in use for a long time. We underline that any such price fortune-telling have sense as ground for price expectations only. Definition (1.1) presents the only trade data price records, the single source of the confirmed price data of executed transactions. Obviously various price expectations in agents minds impact agents decisions to go into particular transaction (1.1). Economic agents with numerous price expectations, value expectations, volume expectations and so on execute market transactions. The trading data of the transactions executed under agents expectations deliver the (1.1) as confirmed price dynamics. We take relations (1.1) as the only tool for option pricing. However, it is clear that agents expectations that approve transactions impact the price properties. Below we show how simple relations (1.1) between the value C , the volume V and the price p of the market transactions and considerations of expectations as factors that impact trading records allow contribute to the BSM, non-linear and stochastic volatility option pricing models.

In Sec.2 we discuss the BSM model taking into account (1.1). In Sec.3 we show that impact of agents expectations can give further complexities of the BSM model. In Sec.4 we discuss non-linear option pricing as result of impact of expectations. In Sec. 5 we discuss Heston’s stochastic volatility model taking into account relations (1.1). In Sec. 6 and 7 we present Discussion and Conclusion. Equation (3.2) defines Section 3 and equation 2.

2. Market transactions and option pricing

Let's take the classical BSM model (Black and Scholes 1973; Merton 1973; Hull 2009) and assume that the underling asset price $p(t)$ follows the standard Brownian motion $dW(t)$ as:

$$dp(t) = p(t) \mu dt + p(t) \sigma dW(t) \quad (2.1)$$

$$\langle dW(t) \rangle = 0; \quad \langle dW(t)dW(t) \rangle = dt \quad (2.2)$$

Here μ – linear trend, σ - dispersion and r – risk-free rate and μ, σ, r – are constant. We use notion $\langle \dots \rangle$ to define averaging of random process. Option price $S(t, p)$ follows the classical BSM equation (2.3):

$$\frac{\partial S}{\partial t} + rp \frac{\partial S}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 S}{\partial p^2} = rS \quad (2.3)$$

Let's maintain all assumptions of the classical BSM model except the main one: we don't take assumptions (2.1, 2.2) on the price $p(t)$. We suppose that stochastic behavior of the price $p(t)$ should be determined by random properties of market transactions with underling assets and in particular by random dynamics of the value $C(t)$ and the volume $V(t)$ of the transactions. Trivial relations define the value $C(t)$ and the volume $V(t)$ of the market transaction $M(t)$ and the price $p(t)$ as:

$$M(t) = (C(t), V(t)); \quad p(t) = \frac{C(t)}{V(t)} \quad (2.4)$$

Relations (2.4) are trivial but they replace initial assumptions (2.1, 2.2) on random properties of the price $p(t)$ by assumptions on random properties of the value $C(t)$ and the volume $V(t)$ of the transaction $M(t)$. To keep simplicity of BSM model as a first approximation let's study the standard Brownian processes (2.2) and take all coefficients like trends, dispersions and rates as constant. Due to (2.4) let's replace assumptions (2.1, 2.2) on the price $p(t)$ by assumptions on the Brownian motion dW_c of the value $C(t)$ and the Brownian motion dW_v of the volume $V(t)$ similar to (2.1):

$$dC(t) = C(t) \mu_c dt + C(t) \sigma_c dW_c(t) \quad (2.5)$$

$$dV(t) = V(t) \mu_v dt + V(t) \sigma_v dW_v(t) \quad (2.6)$$

$$\langle dW_c(t) \rangle = 0; \quad \langle dW_c(t)dW_c(t) \rangle = dt \quad (2.7)$$

$$\langle dW_v(t) \rangle = 0; \quad \langle dW_v(t)dW_v(t) \rangle = dt \quad (2.8)$$

Let's take that Brownian processes dW_c and dW_v are correlated as:

$$\langle dW_c(t)dW_v(t) \rangle = \lambda dt \quad (2.9)$$

These assumptions determine the random behavior of the price $p(t)$ (2.4) as:

$$dp(t) = \frac{dC(t)}{V(t)} - \frac{C(t)}{V(t)} \frac{dV(t)}{V(t)} \quad (2.10)$$

Relations (2.5, 2.7, 2.10) define $dp(t)$:

$$dp(t) = p(t)[(\mu_c - \mu_v) dt + (\sigma_c dW_c(t) - \sigma_v dW_v(t))] \quad (2.11)$$

Two Brownian processes (2.5; 2.6) cause that option price S should depend on time t and two variable and we take them as the price p and the volume V . Many papers present option pricing under action of multiple Brownian processes (Broadie and Detemple 1997; Rapuch and Roncalli 2004; Carmona and Durrleman 2006; Li, Deng and Zhou 2010; Hull and White 1987; Heston 1993; Ball and Roma 1994; Britten-Jones and Neuberger 2000; Cohen and Tegner 2018). We don't aim the derivation of option pricing equation under action of two Brownian processes. For the simplest assumptions of classical BSM model we demonstrate that randomness of the value C and the volume V of market transactions (2.4) induces 2-dimensional option pricing equation. It is easy to show (Hull 2009; Poon 2005) that (2.6-2.9; 2.11) cause:

$$\frac{\partial S}{\partial t} + rp \frac{\partial S}{\partial p} + rV \frac{\partial S}{\partial V} + \frac{1}{2} p^2 \sigma^2 \frac{\partial^2 S}{\partial p^2} + \frac{1}{2} V^2 \sigma_v^2 \frac{\partial^2 S}{\partial V^2} + pV\varrho \frac{\partial^2 S}{\partial V \partial p} = rS \quad (2.12)$$

$$\sigma^2 = \sigma_c^2 - 2\lambda\sigma_c\sigma_v + \sigma_v^2 ; \varrho = \lambda\sigma_c\sigma_v - \sigma_v^2 \quad (2.13)$$

We use risk-free portfolio Π and risk-free rate r .

$$\Pi(t) = S - \alpha p - \beta V ; \quad d\Pi = r\Pi dt \quad (2.14)$$

The derivation of (2.12; 2.13) is standard (Hull, 2009; Poon, 2005) and we omit it here. Equation (2.12) describes option price dynamics on 2-dimensional space (p, V) . If Brownian motion dW_v (2.6) is identical to (2.5) dW_c with (2.9) $\lambda=1$ then only one Brownian process govern random price and (2.11) can be presented as

$$dp(t) = p(t)[(\mu_c - \mu_v) dt + (\sigma_c - \sigma_v) dW_c(t)] \quad (2.15)$$

and option equation (2.12) is reduced to the classical BSM equation (2.3) with σ^2

$$\sigma^2 = (\sigma_c - \sigma_v)^2$$

The same reduction from (2.12) to (2.3) follows if the volume $V(t)$ is a regular function or constant. Thus classical BSM model describes option pricing in the assumption that the value C and the volume V of the market transactions follow the identical Brownian motion or the value C or volume V are regular functions or constant. We omit here the change of variables that leads to the equation as it is simple and gives no new meaning.

If the value $C(t)$ (2.5) and the volume $V(t)$ (2.6) are described by different Brownian processes and the price $p(t)$ follows (2.11) then the option price S should obey the 2-dimensional BSM-like equation (2.12).

3. Expectations and option pricing

As we show above the simple relations (2.4) enlarge the “space” of the BSM equation from one to two dimensions. However the market transactions are performed by economic agents and agents take decisions on the value, the volume and the price of the transactions under personal expectations. Agents expectations approve market transactions and impact evolution of the value, the volume and the price of transactions. Different agents may take their decisions on base of different expectations and random perturbations of numerous expectations may cause random disturbances of the value, the volume and the price of transactions. Studies of expectations and their impact on economic and financial markets have a long history and we mention only some starting with Keynes (1936), Muth (1961) and Lucas (1972) and further research by (Sargent and Wallace 1976; Hansen and Sargent 1979; Kydland and Prescott 1980; Blume and Easley 1984; Greenwood and Shleifer 2014; Manski 2017). Usually expectations are treated as agents forecasts of trends and values of economic and financial variables, inflation and bank rates, income and prices, technology and weather forecasts and etc. We regard agents expectation as their assumptions on future state and dynamics of any economic variables or factors that can impact economic development. Variability and diversity of factors and variables that establish agents expectations make them the major source of the randomness that impact decisions on market transactions and through them the major stochastic impact on price dynamics. Expectations are most ambiguous economic issues. To simplify the problem as much as possible let’s take the following assumptions. Let’s propose that each market transaction is performed under expectations those approve decisions on the value and the volume of transactions taken by two agents involved into transaction. Let’s define as x_j , $j=1,..4$ expectations of agents involved into market transaction. Let’s take that x_1 , x_2 – describe expectations on the value and the volume of the first agent – the seller and x_3 , x_4 describe expectations on the value and the volume of the second agent – the buyer. Let’s assume that expectations impact each other and the value and the volume of the transaction depend on all expectations of both agents. We assume that random changes of expectations dx_j cause random change of the value C and the volume V of the transactions:

$$dC = C \sum_{j=1,4} A_j dx_j \quad ; \quad dV = V \sum_{j=1,4} B_j dx_j \quad (3.1)$$

$$dp = p \sum_{j=1,4} D_j dx_j \quad ; \quad D_j = A_j - B_j \quad ; \quad j = 1,..4 \quad (3.2)$$

$$A_j = \frac{\partial \ln C}{\partial x_j} \quad ; \quad B_j = \frac{\partial \ln V}{\partial x_j} \quad (3.3)$$

Relations (3.3) model the dependence of the value C and the volume V of transactions on expectations x_j . As we discussed above agent's expectations may be based on any economic or financial variables or factors those impact economic development and may determine decisions on making the market transactions with underlying assets. Let's regard the value C and volume V of market transactions and coefficients A_j and B_j as functions of expectations x_j . That imply possible dependence of coefficients (3.3) on agents expectations $x_j, j=1,..4$. As we show below this simple assumption permit study non-linear models for option pricing.

For simplicity let's take that random expectations dx_j follow the standard Brownian motion

$$dx_j = \mu_j dt + \sigma_j dW_j \quad ; \quad j = 1,..4 \quad (3.4)$$

$$< dW_i dW_j > = \lambda_{jk} dt \quad ; \quad \lambda_{jj} = 1 \quad ; \quad |\lambda_{jk}| \leq 1 \quad ; \quad j, k = 1,..4 \quad (3.5)$$

Four Brownian processes imply that option price should depend on four variables and we take price p and three expectations $x_j, j=1,2,3$ (3.4; 3.5) as independent variables. BSM-like equation on option price $S=S(t,p,x_1,x_2,x_3)$ takes form

$$\frac{\partial S}{\partial t} + rp \frac{\partial S}{\partial p} + rx_j \frac{\partial S}{\partial x_j} + \frac{1}{2} p^2 \sigma_p^2 \frac{\partial^2 S}{\partial p^2} + \frac{1}{2} \sum_{j,k=1,2,3} \lambda_{jk} \sigma_j \sigma_k \frac{\partial^2 S}{\partial x_j \partial x_k} + p \varrho_j \frac{\partial^2 S}{\partial x_j \partial p} = rS \quad (3.6)$$

$$\sigma_p^2 = \sum_{j,k=1,..4} \lambda_{jk} D_j D_k \sigma_j \sigma_k \quad (3.7)$$

$$\varrho_j = \sum_{k=1,..4} \lambda_{jk} D_k \sigma_k \sigma_j \quad (3.8)$$

It is clear that the seller and the buyer may perform market transactions with underlying asset on base of numerous expectations. Agents expectations are most uncertain and most influential economic issues that deliver major randomness to financial markets and economics as a whole. Ensemble of these expectations and their disturbances deliver additional uncertainty to asset price $p(t)$ and through it to option pricing. In (Olkhov 2019) we present a simple model that describes direct impact of small fluctuations of expectations on price and return fluctuations and price-volume relations. Similar model can be used to model impact of ensemble of expectations on option pricing.

4. Non-linear option pricing models

Expectations that govern the underlying asset transactions may concern options pricing dynamics. In other words – market traders may perform transactions with underlying assets on base of information and assessments of corresponding option trading data. Actually we believe that professional investors and traders use all market information available to them to establish their expectations and to take the preferable market transaction. Thus the value C and the volume V of market transactions with underlying may depend on agents expectations and coefficients (3.3) may depend on agents expectations formed by current option price S or

its derivatives by time t or by price p and etc. For example, relations (3.3) that describe direct dependence on option price S may model nonlinear dependence of option pricing models. Non-linear option pricing models are studied for more than 25 years (Bensaid, et.al. 1992; Sircar and Papanicolaou 1998; Frey 2008; Frey and Polte 2011; Loeper 2018).

Expectations of investors and traders on option pricing data impact their market transactions on underlying assets and cause non-linear option pricing equations. As a toy model let's regard dependence of the value $C(t)$ of transactions on expectation x that is determined by the option price S

$$C = C(t, x) ; \quad A(t, S) = \frac{\partial \ln C}{\partial x} \quad (4.1)$$

For simplicity let's take that the value $C(t)$ depends on a single expectation x and the volume $dV=0$. Let's assume that investors and traders forecast change dx of their expectation due to standard Brownian motion and

$$dx = \mu dt + \sigma dW ; \quad \langle dW dW \rangle = dt \quad (4.2)$$

Then, due to (2.10; 2.11) change of the price dp can be presented as:

$$dp(t) = pA(S)dx = pA(S)[\mu dt + \sigma dW] ; \quad dV = 0 \quad (4.3)$$

and the equation (2.12) is reduced to classical BSM equation (2.3) with non-linear term and takes the form:

$$\frac{\partial S}{\partial t} + rp \frac{\partial S}{\partial p} + \frac{1}{2} A^2(S) \sigma^2 p^2 \frac{\partial^2 S}{\partial p^2} = rS \quad (4.4)$$

Definite dependence of A^2 on the option price S and on its derivatives should be studied for each particular case separately. We don't study here any particular non-linear BSM-like equation but outline a simple and direct way to take into the consideration the impact of the expectations on the option pricing and method for derivation of corresponding the non-linear BSM-like equations. The similar assumptions can induce more sophisticated non-linear BSM-like equations in two, three or four dimensions that take into account impact of two, three or four expectations starting with (3.6). Such non-linear BSM-like equations can describe the dependence of coefficients (3.3) on the option price S or its derivatives by time t or by price p .

Description of the impact of expectations on transactions and their value C and volume V is rather complex problem. There are numerous agents involved into market transactions with underlying or with options. Different agents perform their transactions under numerous expectations. As we mentioned above agents may establish their expectations on base of any economic and financial variables, market and tax trends, technology and climate forecasts, and on base of any social or psychology factors that may impact agent's mood. Thus

description of option pricing as well as description of asset pricing should take into account definite “mean” action of various expectations that impact decisions of different agents. Such “mean” expectations as well as fluctuations from “mean” expectations impact underlying and option pricing. The methods for description of distribution of expectations of different agents and modeling “mean” expectations are presented in Olkhov (2019). These methods introduce distributions of agents, transactions and expectations that help describe price-volume and return-volume disturbances for asset pricing. The approach to economic modeling developed by (Olkhov 2016a; 2016b) gives opportunity to discuss some hidden problems of option pricing (Olkhov 2016c). We refer for these studies for further details.

5. Stochastic volatility

It is well known that assumption on constant volatility of the classical BSM model doesn't match the market reality. Numerous extensions of BSM equations were proposed to describe impact of stochastic volatility on option pricing. Stochastic volatility models were developed starting at least with Cox and Ross (1976) and then followed by Hull and White (1987), Heston (1993), Ball and Roma (1994), Saikat (1996), Poon (2005), Engle and Figlewski (2014), Cohen and Tegner (2018). Further studies of stochastic volatility models concern usage of various assumptions on properties of stochastic processes that may describe real properties of market volatility variations. Stochastic volatility models (Heston 1993, Poon 2005) describe transition from 1-dimensional BSM equation to 2-dimensional heat-type equation. As we show impact of the value and the volume of transactions with underlying induce 2-dimensional BSM-like equation (2.12; 2.13). If one takes into account impact of expectations those approve decisions on market transactions then option pricing may obey two, three or four-dimensional BSM-like equations with constant volatilities. Extensions of equations (2.12; 2.13) to stochastic volatility model introduce two additional random variables: random value volatility σ_c^2 and random volume volatility σ_v^2 that follow Brownian motion $dW_{\sigma c}$ and $dW_{\sigma v}$. Let's define

$$x = \sigma_c^2 \quad ; \quad y = \sigma_v^2 \quad (5.1)$$

Relations (2.5-2.9) those define equation (2.12) stochastic volatility model are complemented by additional relations (Heston, 1993; Poon, 2005)

$$dx = \alpha_x(\theta_x - x)dt + \sigma_x\sqrt{x} dW_x \quad (5.2)$$

$$dy = \alpha_y(\theta_y - y)dt + \sigma_y\sqrt{y} dW_y \quad (5.3)$$

Relations (2.5-2.9) and (5.1; 5.2) define four independent Brownian motions and induce corresponding 4-dimension BSM-like equation. To avoid excess complexity let's present 3-

dimensional Heston-like equation $S=S(t,p,V,x)$ that model the stochastic value volatility $x=\sigma_c^2$ and keep the volume volatility $y=\sigma_v^2$ - constant.

$$\begin{aligned} \frac{\partial S}{\partial t} + rp \frac{\partial S}{\partial p} + rV \frac{\partial S}{\partial V} + [\alpha_x(\theta_x - x) - \vartheta x] \frac{\partial S}{\partial x} + \frac{1}{2} p^2 (x - 2\lambda\sigma_v\sqrt{x} + \sigma_v^2) \frac{\partial^2 S}{\partial p^2} + \frac{1}{2} V^2 \sigma_v^2 \frac{\partial^2 S}{\partial V^2} + \\ \frac{1}{2} \sigma_x^2 x \frac{\partial^2 S}{\partial x^2} + p\sigma_x\sqrt{x} [\sqrt{x}\lambda_{cx} - \sigma_v\lambda_{vx}] \frac{\partial^2 S}{\partial x \partial p} + V\sigma_v\sigma_x\sqrt{x} \lambda_{vx} \frac{\partial^2 S}{\partial V \partial x} + pV\varrho \frac{\partial^2 S}{\partial V \partial p} = rS \end{aligned} \quad (5.4)$$

$$< dW_x(t)dW_c(t) > = \lambda_{xc} dt \quad ; \quad < dW_x(t)dW_v(t) > = \lambda_{xv} dt \quad (5.5)$$

If dW_v is identical to dW_c or for is V -const then the equation (2.12) is reduced to the classical (2.3) and (5.4) is reduced to the Heston stochastic volatility equation (5.6) (Heston 1993)

$$\frac{\partial S}{\partial t} + rp \frac{\partial S}{\partial p} + [\alpha_x(\theta_x - x) - \vartheta x] \frac{\partial S}{\partial x} + \frac{1}{2} p^2 x \frac{\partial^2 S}{\partial p^2} + \frac{1}{2} \sigma_x^2 x \frac{\partial^2 S}{\partial x^2} + \lambda_{xc}\sigma_x p x \frac{\partial^2 S}{\partial x \partial p} = rS \quad (5.6)$$

During more then 25 years transitions of the option pricing from constant to stochastic volatility models in the Heston approximation were described in numerous papers (Heston 1993; Poon 2005; Cohen and Tegner 2018) and we are not going to reproduce them once more. We just show that description of stochastic volatility of the value and the volume of transactions or stochastic volatility of expectations increase dimension of the equations (2.12) and (3.6) and add extra complexity for option pricing modeling. Nonlinear equations (4.4) with constant volatilities σ^2 also can be starting points for extension to stochastic volatility approximation. Description of stochastic volatility increase dimension of the equation (4.4) and add extra complexity for solving high dimensional nonlinear equations.

6. Discussion

Let's discuss some internal problems of option pricing modeling. Studies of these problems may clarify relations between forecasting of option pricing on base of the BSM-like equations and real market data.

High frequency trading can deliver thousands records of market transactions per second. This information is very useful for short-term market assessment during one hour or one day. But even intraday volatility assessments may require initial aggregation of high frequency market data for seconds, minutes or hours. Option pricing assessments for weeks and months may need time data aggregation during hours or days. Aggregation of market data by time term t_l means that the option price evolution model has internal time scale t_l . This time scale t_l may impact random properties of the underlying price and impact option price model dynamics. The second time scale $t_2 > t_l$ is responsible for the averaging procedure $\langle \dots \rangle$ to assess the mean value and volatility of Brownian processes (1.2). The scale $1/t_2$ define maximum frequency scale for the problem under consideration. Usage of different time scales t_l, t_2 to

solve the same options pricing problem with scale - time to maturity T may cause different properties, as distinctive frequency scales will be different. On the other hand usage of time scales t_1 and t_2 causes aggregation of economic and financial variables of the problem. In particular it means aggregation of the value and the volume of the transactions with underlying during time term t_2 to obtain correct values for price $p(t)$ due to relations (2.10; 2.12). Roughly speaking, one should measure the sum of the value $C(t)$ and sum of the volume $V(t)$ of all transactions during time term t_2 and their ratio (2.4) should define the mean price $p(t)$. Hence the solution $S(T, p, V)$ of the option price equation (2.12) should depend on the internal time scale t_2 .

Existence of the internal time scales $t_1 < t_2$ for the option pricing and the requirement to use relations (2.4, 2.10) to define the price $p(t)$ at moment t arises the problem of impact of expectations on asset and option pricing. Indeed, relations (2.4) for the value $C(t)$ and the volume $V(t)$ aggregated by all transaction during time term t_2 define so-called volume weighted average price $p(t)$ (VWAP) of transactions (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018). However expectations of agents those approve agents decisions on the value of the volume of the performed transactions can be based on any factors that might impact agents will to make a deal. For example, agents expectations can be based on assessment of price $\pi(t)$ as simple average during same time term t_2 . The difference between VWAP and assessments of price $\pi(t)$ as simple average may be great. Thus agents expectations may reflect price assessments based on different models and methods and that induce additional disturbances and stochastic to the price results (1.1) of the market transactions.

Above issues and variety forms and dimensions of possible linear or nonlinear equations (2.12; 3.6; 4.4; 5.4) that can model options pricing in different assumptions on constant or stochastic volatility or on impact of expectations arise the problem of sufficiency requirements of option modeling. The classical BSM equations (2.3) deliver simple 1-dimensional model that describe the option price dynamics with reasonable accuracy. Classical BSM model has only two parameters. One should only wonder how such a simple equation (2.3) gives so good model forecast for option pricing. Any further extensions of the classical BSM model add more accuracy but that cost extra complexity.

We consider the classical BSM model assumptions and replace the only one. We take into account the dependence of underlying price on the value and the volume of the market transactions. This trivial point gives sufficient reasons to extend the 1-dimensional BSM equations (2.3) to 2-dimensional equations (2.12). But such extension even for the case with

constant parameters costs extra volume volatility and correlation coefficients. Transition from constant to stochastic volatility approximations turn the model to much more complex 3- or 4-dimensional equations (5.4).

Common understanding that market transactions are governed by agents expectations and infinite diversity of these expectations transfers the relatively simple 2-dimensional equation (2.12) to 4-dimensional equation (3.6) with constant coefficients. Any attempt to model stochastic volatility of underlying asset price or stochastic volatility of expectations would increase the dimension and the complexity of the equations. Moreover any sophistication of option pricing equations requires additional sophistication of market econometrics. Actually market data on the value and the volume of executed transactions are among the usual and we hope that we don't add excess complexity to option price description on base of (2.12). As well the attempt to take into considerations the impact of agents random expectations on market transactions and use corresponding econometric data may be not too simple.

We regard the balance between the accuracy of the description and complexity of econometrics as main problem for asset and option pricing modeling. We don't see any place for unique "correct and perfect" option pricing model equation but propose that different approximations should serve for different option markets. Any step towards accuracy will be compensated by two steps back to complexity due to changes in agents expectations and their impact on asset and option pricing. Nonlinear relations even in simplest form modeled by (4.4) reflect the tip of the complex mutual dependence between underlying and options.

7. Conclusion

After almost 50 years since publication the classical BSM options pricing model remains the source for further studies. We reconsider the BSM model and consistently use the price definition (1.1) as the only correct price records determined by executed market transactions. The relations (1.1) present the price p as coefficient between the value C and the volume V of the market transaction. The usage of simple relations (1.1) for the BSM option pricing model extends the BSM equation (1.3) from one to two dimensional BSM-like equation (4.1). As we show in Sec.2 the classical BSM one-dimensional model implicitly use the assumption that the value C and the volume V follow the identical Brownian motion. The deviation of the value C and the volume V from identical Brownian processes leads to 2-dimensional BSM-like model (2.12). Any price definitions, models and forecasts different from (1.1) may have sense as ground for agents price expectations only. It is clear that agents price expectations those approve execution of the market transactions impact the price dynamics (1.1).

Nevertheless relations (1.1) remain the only correct set of price records of executed market transactions. The influence of agents price expectations on option pricing lead to further complicating and increase of dimension of the option pricing models. We present simple models that take into account impact of agents expectations on performance of market transactions. Agents may go into transactions under their individual expectations on the value, volume or price dynamics or any other expectations. As we show in Sec. 3 and 4 taking into account agents expectations can further increase the dimension of the BSM model or give further complication for non-linear option model.

Equation (3.6) serves as starting point for modeling (4.4) non-linear relations between underlying and option market and opens the way for equation (5.4) that describe the Heston stochastic volatility model impact on option pricing. The set of the BSM-like equations (2.12; 3.6; 4.4; 5.4) have only common origin: the simple relations (1.1) that determine the price p as the coefficient between the value C and the volume V of the market transactions and assumption that agents go into transactions under their individual expectations.

These equations don't simplify the description of the option pricing but may adopt impact of the real economic and financial factors on the underlying and the option pricing. The most difficult problem of all further extensions is the achieving the balance between the next level of accuracy of the asset and option pricing model and the next level of complexity. Complex market relations require complex equation to describe real market processes and we hope that usage of equations (2.12; 3.6; 4.4; 5.4) and obvious directions for their extensions may improve current asset and option pricing modeling.

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